Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00065-4

J. Appl. Maths Mechs, Vol. 64, No. 3, pp. 425–434, 2000

0021-8928/00/\$—see front matter

THE CONDITION FOR LOW STRESSES IN A PIECEWISE HOMOGENEOUS WEDGE OF NON-LINEARLY ELASTIC MATERIALS†

M. A. ZADOYAN

Yerevan

(*Received* 12 *May* 1999)

The low-stress regions, to the points of which zero stressed states at the edges of the contact surfaces correspond, are constructed in the space of physical and geometrical parameters of a piecewise homogeneous wedge of non-linear elastic materials. When these parameters are specified, one can judge the strength of the joint from these zones. If the values of only some of the parameters are known, the remaining parameters when the edges are being designed can be chosen so that the conditions for low stresses are satisfied. In particular, a three-dimensional low-stress region is constructed when the wedge is made of three materials which are strengthened in accordance with a power law.

The case of linearly elastic wedges made of two materials was considered in [1], and for materials with power-law strengthening in [2, 3].

1. THE GENERAL CASE

We will investigate the stressed state for an arbitrary shear in the neighbourhood of a corner point of a piecewise homogeneous solid, made of n wedge-shaped prisms, the materials of which are strengthened in accordance with the power law

$$\sigma_0 = k\varepsilon_0^m$$

where σ_0 and ε_0 are the stress and strain intensities, the parameter m is assumed to be the same for all the materials, while the strain modulus k is assumed to be different. The angles at the vertices of the wedge components will be denoted by α_i, while the strain moduli of the materials will be denoted by k_i, respectively, where $i = 1, 2, \ldots, n$ (Fig. 1). Quantities in the ranges $A_{i-1} \leqslant \theta \leqslant A_i$, where $A_i = \alpha_i + \alpha_2 + \ldots + \alpha_i, A_0 = 0$, will be given the subscripts i.

The stresses and displacements in these ranges will be sought in the form

$$\begin{aligned} &\tau_{rzi} = \lambda k_i^{(\lambda-1)m} f_i \chi_i, \quad \tau_{rzi} = k_i r^{(\lambda-1)m} f_i' \chi_i \\ &w_i = r^\lambda f_i, \quad \chi_i = (f_i'^2 + \lambda^2 f_i^2)^{(m-1)/2}, \quad i = 1, 2, \ldots, n \end{aligned} \tag{1.1}$$

The system of functions $f_i = f_i(\theta, \lambda)$ defines the eigenfunction, while λ is the eigenvalue of the problem in question. Substituting the stress components from (1.1) into the third equilibrium equation, we arrive at a second-order ordinary differential equation if f_i

$$(f_i'\chi_i)' + \eta f_i \chi_i = 0, \quad \eta = \lambda[1 + (\lambda - 1)m] \tag{1.2}$$

For boundary conditions of the first kind we have

$$f_1'(0) = f_n'(A_n) = 0 \tag{1.3}$$

On the contact surfaces

$$\begin{aligned} &f_i = f_{i+1}, \quad f_i'\chi_i = \delta_i f_{i+1}' \chi_{i+1} \quad \text{when} \quad \theta = A_i \\ &\delta_i = k_{i+1} / k_i, \quad i = 1, 2, \ldots, n-1 \end{aligned} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 3, pp. 444–453, 2000.

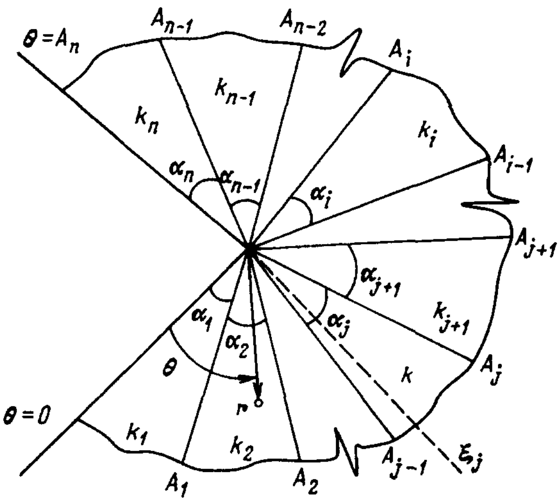

Fig. 1

Introducing the new function $\psi(\theta, \lambda)$

$$f_i' = f_i \psi_i \tag{1.5}$$

from (1.2) we obtain a first-order differential equation

$$\psi_i' = -\frac{(\psi_i^2 + \lambda^2)(\psi_i^2 + \omega^2)}{\psi_i^2 + \lambda^2 p}; \quad \omega^2 = \lambda(\lambda + p - 1), \quad p = \frac{1}{m} \tag{1.6}$$

The conditions on the contact surfaces, by (1.3), will be

$$\mu_i(\mu_i^2 + \lambda^2)^{(m-1)/2} - \delta_i \nu_i (\nu_i^2 + \lambda^2)^{(m-1)/2} = 0, \quad i = 1, 2, \ldots, n-1 \tag{1.7}$$

$$\mu_i = \psi_i(A_i, \lambda), \quad \nu_i = \psi_{i+1}(A_i, \lambda)$$

We will assume that the displacement changes sign inside the range of one of the intermediate wedges ($i = j$). The boundary conditions for Eq.(1.6) will then be

$$\psi_1(0) = \psi_n(A_n) = 0 \tag{1.8}$$

We will represent the general solution of Eq.(1.6) in the following form

$$F(\psi_i) = H_i - \theta, \quad i \neq j; \ i = 1, 2, \ldots, n; \quad j \neq 1, \quad j \neq n \tag{1.9}$$

$$F(\psi_{j1}) = H_j - \theta \ \text{ when } \ A_{j-1} \leq \theta \leq \xi_j$$

$$F(\psi_{j2}) = \xi_j - \theta \ \text{ when } \ \xi_j \leq \theta \leq A_j$$

$$F(x) = \operatorname{arctg}\frac{x}{\lambda} + \frac{1-\lambda}{\omega}\operatorname{arctg}\frac{x}{\omega}, \quad \xi_j = H_j + \frac{1}{2}\Lambda, \quad \Lambda = \pi\left(1 + \frac{1-\lambda}{\omega}\right)$$

Here H_i and H_j are arbitrary constants, and we have also used passages to the limit from the right and left to the point $\theta = \xi$. Introducing the new unknown constants $\varphi = H_i - A_{i-1}$, where $\varphi_1 = 0$, $\varphi_n = \alpha_n$, we obtain

$$F(\mu_i) = \varphi_i - \alpha_i, \quad i = 1, 2, \ldots, n-1; \quad i \neq j \tag{1.10}$$

$$F(\mu_j) = \varphi_j - \alpha_j + \Lambda, \quad F(\nu_i) = \varphi_{i+1}, \quad i = 1, 2, \ldots, n-1 \tag{1.11}$$

Using the last equation and eliminating φ_i in the first equation of (1.10) and (1.11), we arrive at the following system

$$
\begin{aligned}
&F(\mu_1) = -\alpha_1 \\
&F(\nu_{i-1}) - F(\mu_i) = \alpha_i, \quad i \neq j, \quad i = 1, 2, \ldots, n-1 \\
&F(\nu_{j-1}) - F(\mu_j) = \alpha_j - \Lambda, \quad F(\nu_{n-1}) = \alpha_n
\end{aligned}
\tag{1.12}
$$

Equations (1.12), together with (1.7), constitute a system of $2n - 1$ equations with $2n - 1$ unknown constants $\mu_1, \mu_2, \ldots, \mu_{n-1}; \nu_1, \nu_2, \ldots, \nu_{n-1}; \lambda$ which enable one, in principle, for specified values of the parameters, to determine the eigenvalue

$$\lambda = \lambda(\alpha_1, \alpha_2, \ldots, \alpha_n;\ \delta_1, \delta_2, \ldots, \delta_{n-1};\ m)$$

If the point ξ_j lies inside the range of the n-th wedge, i.e. $j = n$, then, taking $\mu_j = \mu_n = 0$ in the first equation of (1.11) we determine $\varphi_n = \alpha_n - \Lambda$. The first two equations of system (1.12) remain unchanged, the last equation is removed, and the penultimate equation takes the form

$$F(\nu_{n-1}) = \alpha_n - \Lambda$$

In the last three equations of (1.9) we must take $H_j = \varphi_n + A_{n-1}$.

Determination of the function f_i. We introduce the following notation

$$\Psi_k(x_1,\ x_2) = \int_{x_1}^{x_2} \psi_k d\theta$$

Integrating Eqs (1.5) and using the conditions for the function f_i on the contact surfaces, we obtain

$$
\begin{aligned}
&f_i = f_1(0)\exp[\Psi_1(0,\ A_1) + \Psi_2(A_1,\ A_2) + \ldots + \Psi_i(A_{j-1},\ \theta)], \quad 1 \le i \le j-1 \\
&f_i = f_n(A_n)\exp[-\Psi_i(\theta,\ A_i) - \Psi_{i+1}(A_i,\ A_{i+1}) - \ldots - \Psi_n(A_{n-1},\ A_n)], \quad j+1 \le i \le n
\end{aligned}
\tag{1.13}
$$

For the j-th wedge we will have

$$
\begin{aligned}
&f_{j1} = f_1(0)\exp[\Psi_1(0,\ A_1) + \Psi_2(A_1,\ A_2) + \ldots + \Psi_{j1}(A_{j-1},\ \theta)], \quad A_{j-1} \le \theta \le \xi_j \\
&f_{j2} = f_n(A_n)\exp[-\Psi_{j2}(\theta,\ A_j) - \Psi_{j+1}(A_j,\ A_{j+1}) - \ldots - \Psi_n(A_{n-1},\ A_n)], \quad \xi_j \le \theta \le A_j
\end{aligned}
\tag{1.14}
$$

If the point ξ_j lies in one of the outermost wedges, for example, in the n-th wedge, the first equations in (1.13) and (1.14) are retained; when $j = n$ the second equation of (1.13) loses its meaning, and instead of the second equation of (1.14) we will have

$$f_{n2} = f_{n2}(A_n)\exp[-\Psi_{n2}(\theta,\ A_n)] \tag{1.15}$$

Hence, the system of functions f_i is determined, apart from two unknown constants $f_1(0)$ and $f_n(A_n)$. One of these can be eliminated by using the obvious matching condition $f_{j1}'(\xi_j) = f_{j2}'(\xi_j)$. To obtain these derivatives, Eq. (1.2) is integrated term by term with respect to θ with $i = j$, initially from A_{j-1} with respect to ξ_j and then from ξ_j with respect to A_j. Further, representing the expressions for f_{j1} and f_{j2} from (1.14) in the form of the product of a constant and an exponential function, and then substituting them into the above matching condition, we obtain

$$
\begin{aligned}
&f_n(A_n) = -f_1(0)\exp[\Psi_1(0,\ A_1) + \Psi_2(A_1,\ A_2) + \ldots + \Psi_{j-1}(A_{j-2},\ A_{j-1}) + \\
&+\Psi_{j+1}(A_j,\ A_{j+1}) + \Psi_{j+2}(A_{j+1},\ A_{j+2}) + \ldots + \Psi_n(A_{n-1},\ A_n)](T_{j1} / T_{j2})^p
\end{aligned}
\tag{1.16}
$$

where

$$T_{j1} = -(\Psi_{j1}N_{j1})_{A_{j-1}} + \eta \int_{A_{j-1}}^{\xi_j} N_{j1} \exp[m\Psi_{j1}(A_{j-1},\ \theta)]d\theta$$

$$T_{j2} = (\Psi_{j2}N_{j2})_{A_j} + \eta \int_{\xi_j}^{A_j} N_{j2} \exp[-m\Psi_{j2}(\theta,\ A_j)]d\theta, \quad N_{ji} = (\psi_{ji}^2 + \lambda^2)^{(m-1)/2}$$

When $j = n$, instead of (1.16) we will have

$$f_{n2}(A_n) = -f_1(0)\exp[\Psi_1(0,\ A_1) + \Psi_2(A_1,\ A_2) + \ldots + \Psi_{n-1}(A_{n-2},\ A_{n-1})]$$

A uniform wedge. If the component wedges are made of the same material, i.e. $\delta_i = 1$, we take $\mu_i = \nu_i$ in Eqs (1.7). From (1.10) and (1.11) we have the relation

$$\varphi_{i+1} = \varphi_i - \alpha_i,\quad i \neq j,\quad i = 1, 2, \ldots, n-1 \tag{1.17}$$

$$\varphi_{j+1} = \varphi_j - \alpha_j + \Lambda \tag{1.18}$$

From (1.13), by specifying the values $i = 1, 2, \ldots j-1$ in succession, we obtain $\varphi_j = -A_{j-1}$. Further, taking $i = n-1, n-2, \ldots j+1$ in succession, from (1.17) we obtain $\varphi_{j+1} = A_n - Aj$. Substituting these expressions into (1.18) and introducing the notation $A_n = 2\pi s$, we obtain $\lambda - 1 = (1-2s)\omega$. For a semi-infinite slit, i.e. for $s = 1$, we have $\lambda - 1 = -p/(p+1)$. This result was first obtained for plane deformation by other methods in [4, 5]. In the case considered, for an arbitrary angle, we obtain (Fig. 2)

$$\lambda = \frac{2 + (p-1)(1-2s)^2 + (1-2s)\sqrt{(p-1)^2(1-2s)^2 + 4p}}{8s(1-s)} \tag{1.19}$$

The formula obtained can also be used in the case of clamped edges by replacing s by $s/2$, and in the case of mixed conditions, by replacing s here by $2s$ [9].

Note that singular stresses at singular points of linearly elastic plane and three-dimensional solids were investigated in [6–8].

The hypersurface of finite stresses. Assuming $\lambda = 1$ in (1.10) – (1.13), defining

$$\mu_i = \operatorname{tg}(\varphi_i - \alpha_i),\quad \nu_i = \operatorname{tg}\varphi_{i+1},\quad i = 1, 2, \ldots, n-1$$

and substituting these expressions into Eq. (1.7), we arrive at a system of $n-1$ equations

$$\operatorname{tg}(\alpha_i - \varphi_i)\,|\cos(\alpha_i - \varphi_i)|^{1-m} + \delta_i \operatorname{tg}\varphi_{i+1}\,|\cos\varphi_{i+1}|^{1-m} = 0 \tag{1.20}$$

containing $n-2$ unknown constants $\varphi_2, \varphi_3, \ldots, \varphi_{n-1}$. After eliminating these parameters, we arrive, in principle, at the equation of hypersurface of finite stresses in the $2n$ space of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n, \delta_1, \delta_2, \ldots, \delta_{n-1}; m$.

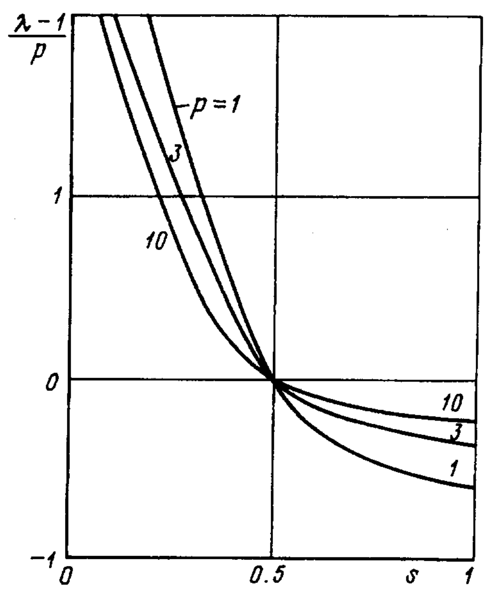

Fig. 2

Linearly elastic materials. Taking $m = 1$ in (1.9), we obtain

$$\psi_i = \lambda \operatorname{tg} \lambda (H_i - \theta), \quad i = 1, 2, \ldots, n-1$$

Further, defining

$$\mu_i = \lambda \operatorname{tg} \lambda (\varphi_i - \alpha_i), \quad \nu_i = \lambda \operatorname{tg} \lambda \varphi_{i+1}$$

and substituting these expressions into Eq. (1.7) with $m = 1$, we obtain

$$\operatorname{tg} \lambda (\alpha_i - \varphi_i) + \delta_i \operatorname{tg} \lambda \varphi_{i+1} = 0, \quad i = 1, 2, \ldots, n-1 \tag{1.21}$$

This is a system of $n - 1$ transcendental equations with $n - 1$ unknown constants $\varphi_2, \varphi_3, \ldots, \varphi_{n-1}, \lambda$. After eliminating the parameters φ_i we obtain an equation in $\lambda = \lambda(\alpha_1, \alpha_2, \ldots, \alpha_n; \delta_1, \delta_2, \ldots, \delta_{n-1})$.

2. THE CASE $n = 2$

We introduce the notation $\alpha_1 = \alpha$, $\alpha_2, = \beta$. In this case, assuming $n = j = 2$, we obtain from (1.9)

$$\begin{aligned} &F(\psi_1) = -\theta, \quad 0 \le \theta \le \alpha \\ &F(\psi_2) = \alpha + \beta - \Lambda - \theta, \quad \alpha \le \theta \le \xi \\ &F(\psi_{22}) = \alpha + \beta - \theta, \quad \xi \le \theta \le \alpha + \beta, \quad \xi = \alpha + \beta - 1/2\Lambda \end{aligned} \tag{2.1}$$

Further, from (1.10) and (1.11), taking $\mu_2 = 0$ and putting $\mu_1 = \mu$ and $\nu_1 = \nu$, we obtain the equations

$$F(\mu) = -\alpha, \quad F(\nu) = \beta - \Lambda \tag{2.2}$$

which, together with the equation

$$\mu(\mu^2 + \lambda^2)^{(m-1)/2} - \delta\nu(\nu^2 + \lambda^2)^{(m-1)/2} = 0 \tag{2.3}$$

comprise a system of equations which define the eigenvalue $\lambda = \lambda(\alpha, \beta, \delta, m)$. In $\alpha\beta\lambda$ space it defines a family of surfaces which depend on the parameters δ and m. This surface is represented in Fig. 3 for $\delta = 2$ and $p = 3$.

In the limiting case when $\xi \to \alpha$, i.e. when $\mu \to -\infty$, $\nu \to -\infty$, it follows from (2.2) and (2.3) that

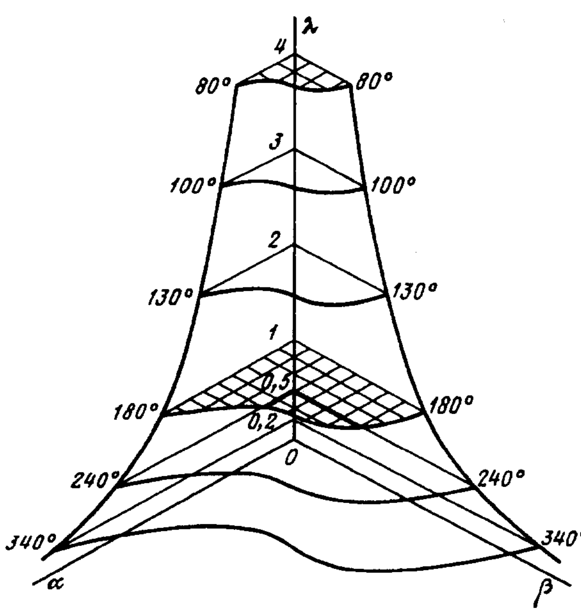

Fig. 3

$$\alpha = \beta = \frac{\pi}{2}\left(1 + \frac{1-\lambda}{\sqrt{\lambda(\lambda + p - 1)}}\right) \tag{2.4}$$

This means that in this case the piecewise homogeneous wedge behaves as a uniform solid. When $\alpha = \pi s$, Eq. (2.4) reduces to (1.19).

For a linearly elastic material ($m = 1$) it follows from system of equations (2.2) – (2.4) that

$$\lambda = \pi/(2\alpha) \ \text{ when } \ \alpha = \beta$$

$$\varphi(\lambda;\ \alpha,\ \beta) = \operatorname{tg}\lambda\alpha + \delta\operatorname{tg}\lambda\beta = 0 \ \text{ when } \ \alpha \neq \beta \tag{2.5}$$

For specified values of δ we will consider λ in (2.5) as an implicit function of α and β. Further, we have

$$d\lambda = \lambda'_\alpha d\alpha + \lambda'_\beta d\beta$$

where the primes denote partial derivatives. When $\alpha = \beta$ we have $\lambda'_\alpha = \lambda'_\beta = -\pi/(2\alpha^2) < 0$, and when $\alpha \neq \beta$ we obtain from (2.5)

$$\lambda'_\alpha = -\varphi'_\alpha/\varphi'_\lambda = -\frac{\lambda}{l\cos^2\lambda\alpha} < 0, \quad \lambda'_\beta = -\varphi'_\beta/\varphi'_\lambda = -\frac{\lambda\delta}{l\cos^2\lambda\beta} < 0,$$

$$l = \frac{\alpha}{\cos^2\lambda\alpha} + \frac{\delta\beta}{\cos^2\lambda\beta}$$

This implies that $d\lambda < 0$, i.e. when α increases, when β = const, or when β increases, when α = const, or when α and β increase simultaneously, λ decreases monotonically.

Taking $\lambda = 1$, from (2.2) and (2.3) we arrive at the equation of the hypersurface of finite stresses [2]

$$\operatorname{tg}\alpha\,|\cos\alpha|^{1-m} + \delta\operatorname{tg}\beta\,|\cos\beta|^{1-m} = 0 \tag{2.6}$$

This surface is a trace in the $\alpha\beta$ coordinate plane (Fig. 4). This is family of limiting curves of finite stresses separating the low-stress zones from the zones of intense stress concentration.

Taking $n = j = 2$, we obtain from the first equations of (1.13) and (1.14), and also from (1.15)

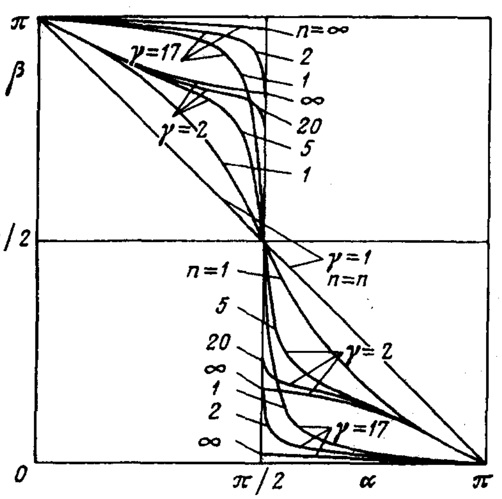

Fig. 4

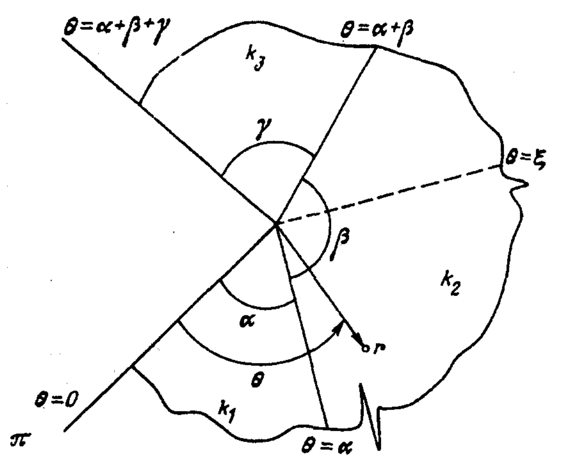

Fig. 5

$$f_1 = Q\exp\Psi_1(0,\ \theta),\quad f_{21} = Q\exp[\Psi_1(0,\ \alpha) + \Psi_{21}(\alpha,\ \theta)]$$

$$f_{22} = -Q\exp[\Psi_1(0,\ \alpha) - \Psi_{22}(\theta,\ \alpha+\beta)](T_{21}/T_{22})^p,\quad Q = f_1(0)$$

The expression for f_{22} refines the corresponding formula in [2, 9].

3. THE CASE $n = 3$

We will introduce the notation $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ (Fig. 5). We will assume that the displacement changes sign in the range corresponding to the central wedge, i.e. $j = 2$. Putting $\varphi_2 = \varphi$, we obtain from Eqs (1.10) and (1.11)

$$\begin{aligned} &F(\mu_1) = -\alpha, \quad F(\nu_1) = \varphi \\ &F(\mu_2) = \varphi - \beta + \Lambda, \quad F(\nu_2) = \gamma \end{aligned} \tag{3.1}$$

These equation, together with the equations

$$\mu_i(\mu_i^2 + \lambda^2)^{(m-1)/2} - \delta_i\nu_i(\nu_i^2 + \lambda^2)^{(m-1)/2} = 0,\quad i = 1,\ 2 \tag{3.2}$$

which follow from (1.7), constitute a system of six equations in the unknown constants μ_1, μ_2, ν_1, ν_2, φ, λ,which define the eigenvalue

$$\lambda = \lambda(\alpha,\ \beta,\ \gamma,\ \delta_1,\ \delta_2,\ m)$$

For a uniform wedge, taking $\delta_1 = \delta_2 = 1$ and $\mu_i = \nu_i$, from (3.1) and (3.2) we obtain the equation

$$\alpha + \beta + \gamma = \Lambda$$

which also leads to formula (1.19).

The hypersurface of finite stresses. When $\lambda = 1$, finding μ_i and ν_i from (3.1) and substituting into (3.2), while taking $i = 1$ and $i = 2$ from (1.20), we arrive at the following equations

$$\begin{aligned} &\operatorname{tg}\alpha\,|\cos\alpha|^{1-m} + \delta_1\operatorname{tg}\varphi\,|\cos\varphi|^{1-m} = 0 \\ &\operatorname{tg}(\beta-\varphi)\,|\cos(\beta-\varphi)|^{1-m} + \delta_2\operatorname{tg}\gamma\,|\cos\gamma|^{1-m} = 0 \end{aligned} \tag{3.3}$$

which contain the unknown parameter, φ. In $\alpha\beta\gamma$ three-dimensional coordinate space this system of equations, for specified values of the parameters δ_1, δ_2 and m, defines the limiting surface of finite stresses (Fig. 6). This surface cuts off, from the coordinate axes, sections equal to π, and leaves traces on the coordinate planes.

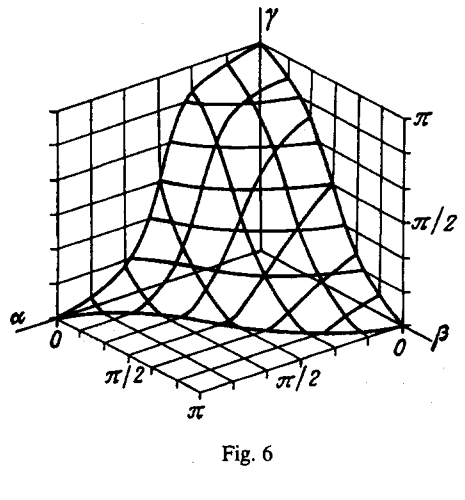

Fig. 6

The limiting surface (3.3) separates a three-dimensional low-stress region (below the surface) from the region of intense stress concentration (above the surface). In other words, the low-stress region for the edge of the contact surfaces of the composite wedge in question will be a three-dimensional region, bounded by surface (3.3) and the coordinate planes containing the origin of coordinates.

We can determine the traces of the limiting surface on the coordinate planes. Assuming $\gamma = 0$, from the second equation of (3.3) we obtain $\varphi = \beta + \pi q$, where q is an integer. Substituting the value of φ into the first equation, we obtain

$$\operatorname{tg}\alpha\,|\cos\alpha|^{1-m} + \delta_1 \operatorname{tg}\beta\,|\cos\beta|^{1-m} = 0$$

This equation defines a family of limiting curves – the traces of surface (3.3) in the $\alpha\beta$ plane. Assuming $\beta = 0$ in (3.3) and eliminating the expressions $\operatorname{tg}\varphi|\cos\varphi|^{1-m}$, we arrive at the equation

$$\operatorname{tg}\alpha\,|\cos\alpha|^{1-m} + \delta_1\delta_2 \operatorname{tg}\gamma\,|\cos\gamma|^{1-m} = 0$$

which defines the traces of the surface (3.3) in the $\alpha\gamma$ plane. Further, assuming $\alpha = 0$, from the first equation of (3.3) we obtain $\varphi = \pi q$. Then, the second equation is converted to the form

$$\operatorname{tg}\beta\,|\cos\beta|^{1-m} + \delta_2 \operatorname{tg}\gamma\,|\cos\gamma|^{1-m} = 0$$

it represents the traces of the limiting surface in the $\beta\gamma$ plane.

For a uniform wedge, i.e. when $\delta_1 = \delta_2 = 1$, system of equations (3.3) is satisfied if we put $\alpha = -\varphi + \pi q_1$ and $\beta - \varphi = -\gamma + \pi q_2$, where q_1 are integers. Eliminating φ, we arrive at the equation of the plane

$$\alpha + \beta + \gamma = \pi \tag{3.4}$$

equally inclined to the coordinate axes and cutting out from the latter sections equal to π. When the plane (3.4) intersects the hypersurface (3.3), three-dimensional regions are separated, to the points of which there correspond low-stress states if the common aperture angle of the wedge $\alpha + \beta + \gamma < \pi$, and also regions to the points of which there correspond intense stress concentrations, if $\alpha + \beta + \gamma > \pi$.

The traces of the surface (3.3) in planes parallel to the coordinate planes are also characteristic limiting curves. Thus, assuming $\gamma = \pi/2$, we conclude from the second equation of (3.3) that $\beta - \varphi = -\pi/2$. Further, noting that α and β vary in the square $0 < \alpha, \beta \leqslant \pi/2$, we obtain from the first equation

$$\sin\alpha\sin^m\beta - \delta_1\cos\beta\cos^m\alpha = 0$$

This equation defines the limiting curves in the $\gamma = \pi/2$ plane. When $\delta_1 = 1$ the limiting line becomes the straight line $\alpha + \beta = \pi/2$. For $m = 1$, this is obvious, and for $m < 1$ it is confirmed by a check.

Assuming $\alpha = \pi/2$ in the first equation of (3.3), we determine $\varphi = -\pi/2$. Taking into account the fact that $0 \leqslant \beta, \gamma \leqslant \pi//2$, we obtain the equation

$$\cos\beta\cos^m\gamma - \delta_2\sin\gamma\sin^m\beta = 0$$

which defines the traces of the limiting surface in the $\alpha = \pi/2$ plane. When $\delta_2 = 1$, this limiting line becomes the straight line $\beta + \gamma = \pi/2$.

Finally, assuming $\beta = \pi/2$ in the second equation of (3.3), taking into account the fact that $0 \leqslant \alpha, \gamma \leqslant \pi/2$, and eliminating $\operatorname{tg}\varphi$. we have

$$\sin 2\varphi = -2\xi_m, \qquad \xi_m = \left(\frac{\delta_2}{\delta_1}\,\frac{\sin\alpha}{\cos^m\alpha}\,\frac{\sin\gamma}{\cos^m\gamma}\right)^{1/(1-m)}$$

Further, evaluating $\operatorname{tg}\varphi|\cos\varphi|^{1-m}$, from the first equation of system (3.3) we obtain, after reduction, the equation

$$\sin\alpha\sin^{1/m}\gamma - \frac{\delta_1}{\delta_2^{1/m}}\cos\gamma\cos^m\alpha\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \xi_m^2}\right)^{(1-m^2)/(2m)} = 0$$

which defines the traces of the surface (3.3) in the $\beta = \pi/2$ plane. When $m = 1$ it is simplified considerably and becomes

$$\sin\alpha\sin\gamma - (\delta_1/\delta_2)\cos\alpha\cos\gamma = 0$$

When $\delta_1 = \delta_2$ the limiting line considered becomes the straight line $\alpha + \beta = \pi/2$. Analysing the graphs it is found that $\alpha + \beta > \pi/2$ when $\delta_1 > \delta_2$ and $\alpha + \gamma < \pi/2$ when $\delta_1 < \delta_2$.

Determination of the function f_i. Taking $n = 3$ and $j = 2$, we obtain from Eqs (1.13) and (1.14)

$$f_1 = Q\exp\Psi_1(0,\theta), \quad f_{21} = Q\exp[\Psi_1(0,\alpha)+\Psi_{21}(\alpha,\theta)]$$

$$f_{22} = -Q\exp[\Psi_1(0,\alpha)-\Psi_{22}(\theta,\alpha+\beta)]\left(\frac{T_{21}}{T_{22}}\right)^p$$

$$f_3 = -Q\exp[\Psi_1(0,\alpha)+\Psi_3(\alpha+\beta,\theta)]\left(\frac{T_{21}}{T_{22}}\right)^p$$

Linearly elastic materials. Taking $m = 1$ and $i = 1, 2$, we obtain from (3.1) and (3.2)

$$\operatorname{tg}\lambda\alpha + \delta_1\operatorname{tg}\lambda\varphi = 0, \quad \operatorname{tg}\lambda(\beta-\varphi) + \delta_2\operatorname{tg}\lambda\gamma = 0$$

Further, eliminating tg $\lambda\varphi$ from this system, we arrive at the following transcendental equation

$$\operatorname{tg}\lambda\alpha + \delta_1\operatorname{tg}\lambda\beta + \delta_1\delta_2\operatorname{tg}\lambda\gamma - \delta_2\operatorname{tg}\lambda\alpha\operatorname{tg}\lambda\beta\operatorname{tg}\lambda\gamma = 0 \tag{3.5}$$

which determines λ for specified values of the parameters α, β, γ, δ_1, δ_2.

If one of the angles of the component wedges is equal to zero or one of the parameters δ_i is equal to unity, a wedge of three different materials reduces to a wedge of two different materials, for which it is known [1] that λ does not have complex values. In the case considered it is necessary to investigate the complex roots of Eq. (3.5).

For the same angles of the component wedges, i.e. when $\alpha = \beta = \gamma$, Eq. (3.5) reduces to the form

$$\operatorname{tg}\lambda\alpha(1+\delta_1+\delta_1\delta_2-\delta_2\operatorname{tg}^2\lambda\alpha) = 0$$

It can be shown that the least positive value of λ will be

$$\lambda = \alpha^{-1}\operatorname{arctg}[(1+\delta_1+\delta_1\delta_2)/\delta_2]^{1/2} \tag{3.6}$$

Here, finally, we have the condition $3\alpha \leqslant 2\pi$. For identical materials, i.e. when $\delta_1, = \delta_2 = 1$, we have $\lambda = \pi/(3\alpha)$.

In the case of a semi-infinite slit ($\alpha = 2\pi/3$), when the material of the central wedge is very rigid ($\delta_1 \to \infty$), from (3.6) we obtain $\lambda = 3/4$.

The condition $\lambda = 1$ ensures a finite stress state at the edge of the contact surfaces considered. From (3.6) we obtain the limiting value

$$\alpha_* = \operatorname{arctg}[(1+\delta_1+\delta_1\delta_2]/\delta_2]^{1/2}$$

When $\delta_1 \to \infty$ we obtain $\alpha_* = \pi/2$.

The limiting values of ξ. For a wedge of three different materials, when $j = 2$ it follows from the first equation of (1.12) that $\xi = \alpha + \varphi + \Lambda/2$, where we have dropped the subscript 2 on ξ. When $\xi \to \alpha$, i.e. when $\mu_1 \to -\infty$, $\nu_1 \to -\infty$ from (3.1) we find $\alpha = \Lambda/2$, and we obtain the system of equations

$$\begin{gathered} F(\mu_2) = -\beta + \Lambda/2, \quad F(\nu_2) = \gamma \\ \mu_2(\mu_2^2+\lambda^2)^{(m-1)/2} - \delta_2\nu_2(\nu_2^2+\lambda^2)^{(m-1)/2} = 0 \end{gathered} \tag{3.7}$$

This means that for given α, β and γ for the first wedge (α) the value of λ is given by the formula for a uniform wedge (1.19), where $s = \alpha/\pi$, while the second and third wedges are deformed together as a piecewise homogeneous wedge of two different materials with corresponding values of $\lambda = (\beta, \gamma, \delta_2, m)$ to be found from system of equations (3.7).

When $\xi \to \alpha + \beta$, i.e. when $\mu_2 \to \infty$, $\nu_2 \to \infty$, we arrive at a similar conclusion from system of equations (3.1) and (3.2): the third wedge (γ) operates as a uniform wedge while the first and second together act as a piecewise homogeneous wedge of two different materials. An analysis of the equations and formulae obtained shows that the low-stress regions do not depend on which constituent wedge the point lies inside (see also [9]).

Note that these investigations can also be carried out without assuming that the displacement is of alternating sign.

I wish to thank S. M. Sarkisyan and K. S. Tadevosyan for their help with the numerical and graphical work.

REFERENCES

1. CHOBANYAN, K. S., *Stresses in Piecewise Homogeneous Elastic Bodies*. Izd. Akad. Nauk ArmSSr, Yerevan 1987.
2. ZADOYAN, M. A., Longitudinal shear of a piecewise homogeneous wedge. *Dokl. Akad, Nauk SSSR*, 1987, **296**, 2, 297–302.
3. ZADOYAN M. A. Low-stress level of a compound wedge under steady creep.
4. CHEREPANOV, G. P., *The Mechanics of Brittle Fracture*. Nauka, Moscow, 1974.
5. RICE J., Mathematical analysis in the mechanics of fracture. In *Fracture*, Vol. 2. Academic Press, New York, 1968, 191–311.
6. MATVEYENKO, V. P., A method for the numerical analysis of a stress singularity at the corner points of three-dimensional bodies. *Izv. Ross. Akad, Nauk. MTT*, 1995, 5, 71–77.
7. BERZENKOV, S. M. and MATVEYENKO, V. P., Semianalytical singular elements for plane and three-dimensional problems of the theory of elasticity. *Izv. Ross. Akad. Nauk. MTT*, 1995, 6, 48–61.
8. BERZENKOV, S. M. and MATVEYENKO, V. P., The optimization of elastic bodies in the neighbourhood of singular points. *Izv. Ross. Akad. Nauk. MTT*, 1996, 2, 93–100.
9. ZADOYAN, M. A., *Three-dimensional Problems of the Theory of Plasticity*. Nauka, Moscow, 1992.

Translated by R.C.G.